JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **127**, 459–469 (1987)

Multiple Periodic Solutions for First-Order Ordinary Differential Equations

GIOVANNI VIDOSSICH

S.I.S.S.A., Strada Costiera 11, 34014 Trieste, Italy

Submitted by V. Laksmikantham

Received November 15, 1985

The last decade has witnessed a tremendous development in the study of the multiplicity of solutions for nonlinear elliptic equations, with many nice results obtained by combining the techniques of upper and lower solutions with the comparison property of eigenvalues of the linear part.

The purpose of this paper is to prove two theorems in the same spirit for periodic solutions to first-order ordinary differential equations of the type

$$x' + q(t)x = f(t, x), \tag{P}$$

where q is a p-periodic function in L^1_{loc} and f is p-periodic in t and satisfies suitable continuity assumptions. There is a great difference between the elliptic case and the present one, due to the fact that the operator

$$Lx = x' + q(t)x$$

is not symmetric in the subspace of L^2 made of periodic C^1-functions. Therefore the usual "symmetric" techniques available for second-order elliptic problems at resonance or not at resonance do not work in our case. However, the different way of comparing eigenvalues used in Vidossich [4] can be suitably adapted to handle the present case. Roughly speaking, we prove results similar to the well-known theorems of Ambrosetti and Mancini [1] and Ambrosetti and Prodi [2] on elliptic equations—however, with much more relaxed assumptions (and different proofs).

We shall use the following well-known lemmas.

LEMMA 0.1. *The eigenvalue problem*

$$x' + q(t)x = \lambda x, \qquad x \ p\text{-periodic} \tag{L}$$

0022-247X/87 $3.00

has exactly one eigenvalue λ_q *and it is defined by*

$$\lambda_q = \frac{1}{p} \int_0^p q(s)\, ds.$$

Therefore it depends continuously on q *in the* L^1*-norm and the following comparison property holds*:

$$q_1 \leqslant q_2 \quad \textit{with} \quad \operatorname{meas}\{q_1 < q_2\} \neq 0 \Rightarrow \lambda_{q_1} < \lambda_{q_2}.$$

LEMMA 0.2. *Let* α, β *be absolutely continuous, p-periodic functions such that* $\alpha \leqslant \beta$ *and either*:

(a) $\alpha' + q(t)\alpha \leqslant f(t, \alpha)$, $\beta' + q(t)\beta \geqslant f(t, \beta)$ *for a.e.t.*; *or, alternatvely*:

(b) $\alpha' + q(t)\alpha \geqslant f(t, \alpha)$, $\beta' + q(t)\beta \leqslant f(t, \beta)$ *for a.e.t.*

Then (P) *has at least one p-periodic solution* x *such that* $\alpha \leqslant x \leqslant \beta$.

The first result is the counterpart of the first theorem in Ambrosetti and Mancini [1].

THEOREM 1. *Let* λ_q *be the unique eigenvalue of* (L). *Assume that* $f(t, 0) \equiv 0$ *and that* $(\partial/\partial x) f$ *exists continuously in* $\mathbb{R}^2$. *If there exists* $A, B > 0$ *and* v *such that one of the following conditions hold*:

(a) $v < \lambda_q$ *and* $(\partial/\partial x) f(t, x) \geqslant \lambda_q$ *for* $|x| \leqslant A$, $(\partial/\partial x) f(t, x) \leqslant v$ *for* $|x| \geqslant B$;

(b) $v > \lambda_q$ *and* $(\partial/\partial x) f(t, x) \leqslant \lambda_q$ *for* $|x| \leqslant A$, $(\partial/\partial x) f(t, x) \geqslant v$ *for* $|x| \geqslant B$;

then (P) *has at least three solutions with one strictly positive and one strictly negative. If, moreover,* $(\partial/\partial x) f(t, \cdot)$ *is strictly monotone of the same type for each* t *on* $]-\infty, 0]$ *and* $[0, +\infty[$, *respectively, then* (P) *has exactly three solutions with one strictly positive and one strictly negative.*

To prove this result, we need first two lemmas.

LEMMA 1. *If there exist* $A, B > 0$ *such that one of the following conditions holds*:

(a) $f(t, x)\, x \geqslant \lambda_q x^2$ *for* $|x| \leqslant A$ *and* $f(t, x)\, x \leqslant \lambda_q x^2$ *for* $|x| \geqslant B$; *or*:

(b) $f(t, x)\, x \leqslant \lambda_q x^2$ *for* $|x| \leqslant A$ *and* $f(t, x)\, x \geqslant \lambda_q x^2$ *for* $|x| \geqslant B$;

then (P) *has at least two nontrivial solutions, one of which is strictly positive and one strictly negative.*

Proof. Assume (a), the case (b) being treated in the same way. Let u be an eigenfunction to (L) for $\lambda = \lambda_q$. It follows from the uniqueness of

Cauchy problems for linear equations that u can be taken strictly positive, and we do. Let ε, $\delta > 0$ be such that

$$\varepsilon u \leqslant A \qquad \text{and} \qquad \delta u \geqslant \max\{A, B\}.$$

Setting $\alpha = \varepsilon u$ and $\beta = \delta u$, from (a) we have

$$\alpha' + q(t)\alpha = \lambda_q \alpha \leqslant f(t, \alpha)$$

$$\beta' + q(t)\beta = \lambda_q \beta \geqslant f(t, \beta).$$

Then Lemma 0.2 guarantees that there exists a p-periodic solution to (P) included between α and β, hence a strictly positive one. Considering $-\beta \leqslant -\alpha$ we get, again from Lemma 0.2, a strictly negative p-periodic solution. Q.E.D.

LEMMA 2. *Let u, v be two different p-periodic solutions to* (P) *such that $u \leqslant v$. Assume that $(\partial/\partial x) f(t, x)$ exists for $0 \leqslant t \leqslant p$, $u(t) \leqslant x \leqslant v(t)$, and satisfies the generalized Caratheodory assumptions. If $(\partial/\partial x) f(t, \cdot)$ is strictly increasing* (*resp. decreasing*) *for each t, then there is no solution x to* (P) *such that $u \leqslant x \leqslant v$, and $u \neq x \neq v$.*

Proof. Assume there is a third p-periodic solution w included between u and v. It is easily seen that we have

$$f(t, x) - f(t, u(t)) = h(t, x)(x - u(t)) \tag{1}$$

with

$$h(t, x) = \int_0^1 f_x(t, \xi(x - u(t)) + u(t))\, d\xi.$$

Consider the semi-linear problems

$$x' + q(t)x = h_i(t)x, \qquad x \text{ } p\text{-periodic} \tag{2}$$

with $h_1(t) = h(t, w(t))$ and $h_2(t) = h(t, v(t))$. From (1) it follows that the two functions

$$w_1 = w - u, \qquad w_2 = v - u$$

are solutions of (2) for $i = 1$ and $i = 2$, respectively. Now let us compare the eigenvalues λ', λ'' of the two linear problems

$$x' + \{q(t) - h_1(t)\}x = \lambda' x, \qquad x \text{ } p\text{-periodic}$$

$$x' + \{q(t) - h_2(t)\}x = \lambda'' x, \qquad x \text{ } p\text{-periodic.}$$

Suppose $(\partial/\partial x) f(t, \cdot)$ is strictly decreasing, the argument being the same as when $(\partial/\partial x) f(t, \cdot)$ is increasing. It follows $-h_1 \leqslant -h_2$ and meas$\{-h_1 < -h_2\} > 0$. Then Lemma 0.1 guarantees that $\lambda' < \lambda''$. But since w_1 and w_2 are nonnull solutions to (2), we have $\lambda' = \lambda'' = 0$, a contradiction. Q.E.D.

Proof of Theorem 1. First, we show that f satisfies the hypothesis of Lemma 1. Assume (a), the proof in case (b) being similar. It follows from the mean value theorem that $f(t, x)x \geqslant \lambda_q x^2$ for $|x| \leqslant A$. By integrating $(d/d\xi) f(t, \xi x)$ it is easily seen that we have

$$f(t, x) = p(t, x)x$$

with

$$p(t, x) = \int_0^1 f_x(t, \xi x)\, d\xi.$$

By continuity, there exists $c > 0$ such that $|f_x(t, x)| \leqslant c$ for $|x| \leqslant A$. For each x with $|x| \geqslant B$ we have

$$\begin{aligned} p(t, x) &= \int_0^{B/|x|} f_x(t, \xi x)\, d\xi + \int_{B/|x|}^1 f_x(t, \xi x)\, d\xi \\ &\leqslant c \frac{B}{|x|} + v\left(1 - \frac{B}{|x|}\right). \end{aligned}$$

Therefore

$$f(t, x)x \leqslant \lambda_q x^2$$

for $|x|$ sufficiently large. This show that f fulfills the assumptions of Lemma 1. It follows from it that (P) has a positive and a negative solution u^+ and u^-. Obviously $u_0 \equiv 0$ is a solution. This shows that there are at least three solutions. To prove the last portion of the theorem, we observe that each pair of solutions are comparable by virtue of uniqueness for Cauchy problems. Therefore if there exists two positive periodic solutions u_1 and u_2, we may assume $u_1 < u_2$. Applying Lemma 2 to the pairs u_0, u_2 we derive a contradiction that shows the uniqueness of the positive solutions when $(\partial/\partial x) f$ fulfills the monotonicity condition. In the same way we state the uniqueness of the negative solution. Q.E.D.

The next result is an Ambrosetti-Prodi-type alternative, cf. [2], with more relaxed assumptions.

THEOREM 2. *Assume that q is continuous, that $f(t, 0) \equiv 0$ and that $(\partial^2/\partial x^2) f$ exists, is continuous, and has constant sign in $\mathbb{R}^2$. Assume, moreover, that the limits*

$$l_-(t) = \lim_{x \to -\infty} \frac{\partial}{\partial x} f(t, x), \qquad l_+(t) = \lim_{x \to +\infty} \frac{\partial}{\partial x} f(t, x)$$

exist uniformly on t, are finite, and satisfy the condition

$$\min\{l_-(t), l_+(t)\} < \lambda_q < \max\{l_-(t), l_+(t)\},$$

where λ_q is the unique eigenvalue of (L). *Then in the space C_p^0 of p-periodic continuous functions there exists a closed connected C^1-manifold M of codimension* 1 *such that $C_p^0 \setminus M$ has exactly two connected components A_1, A_2 and for the boundary value problem*

$$x' + q(t)x = f(t, x) + h(t), \qquad x \text{ p-periodic}, \tag{3}$$

we have the following:

(a) *if $h \in A_1$, then* (3) *has no solution*;

(b) *if $h \in A_2$, then* (3) *has exactly two solutions*;

(c) *if $h \in M$, then* (3) *has exactly one solution.*

Moreover, $A_2 \cup M$ is closed and convex.

It could further be shown that $A_2 \cup M$ is unbounded "below" in the sense that

$$h \in A_2 \cup M \text{ and } g \in C_p^0,\ g \geq 0 \Rightarrow h - g \in A_2 \cup M.$$

In fact, the solution corresponding to h is an upper (resp. lower) solution for the problem corresponding to $h - g$, while a lower (resp. upper) solution can be constructed by choosing a positive solution u to (L) for $\lambda = \lambda_q$ and then setting $\alpha = -cu$ (resp. $\beta = cu$) with $c > 0$ sufficiently large.

Proof of Theorem 2. We shall adapt the original argument of Ambrosetti and Prodi [1] to our nonsymmetric problem, the main technical tool in this respect being furnished by a systematic use of the comparison for eigenvalues due to Lemma 0.1. Consider the mapping

$$T\colon C_p^1 \to C_p^0$$

defined by

$$Tx(t) = x'(t) + q(t)\,x(t) - f(t, x(t)).$$

Since q and $(\partial^2/\partial x^2) f(t, x)$ are continuous, T is of class C^2, and the derivatives of T at u are given by

$$T'(u) \cdot x(t) = x'(t) + q(t)x(t) - \frac{\partial}{\partial x} f(t, u(t)) \cdot x(t) \tag{4}$$

$$T''(u) \cdot v \cdot w(t) = \frac{\partial^2}{\partial x^2} f(t, u(t)) \cdot v(t) \cdot w(t). \tag{5}$$

Below we shall prove the following facts:

(A) T is a proper mapping (i.e., $T^{-1}(K)$ is compact whenever K is);

(B) the singular set W or T is non-empty, closed, connected, and every point of W is an ordinary singular point (in the terminology of Ambrosetti and Prodi [2]);

(C) if $h \in T(W)$, then the given boundary value problem has a unique solution.

Assume for the moment (A), (B), and (C) and let us prove Theorem 2. We set

$$M = T(W)$$

and hence conclusion (c) holds. Since W is closed and connected and T is proper, M is closed and connected. Since all points of W are ordinary, it follows from Theorem 2.7 of Ambrosetti and Prodi [2] that M is a manifold of codimension 1. Thus by Proposition 2.5 of Ambrosetti and Prodi [2], we can say that $C_p^0 \backslash M$ has at most two connected components. Since T is proper, from Proposition 1.5 of Ambrosetti and Prodi [2] it follows that the number of solutions to the equation

$$T(u) = g$$

is constant when g belongs to the same connected component. To compute this number, choose $u_0 \in W$. By Theorem 2.11 of Ambrosetti and Prodi [2], we can compute locally the number of solutions of the equation $T(u) = g$ when g lies on a segment which is transversal to M at $T(u_0)$. These solutions are 2 or 0 according to the side of M on which g lies. Therefore the connected components of $C_p^0 \backslash M$ are exactly 2, and the equation $T(u) = g$ has no solution or 2 solutions, according to which component g belongs. Then also (a) and (b) hold. The set $T(C_p^0) = A_2 \cup M$ is closed by (A). To show that $A_2 \cup M$ is convex, suppose for the moment $(\partial^2/\partial x^2) f > 0$. Then f is convex. Therefore for any $h_1, h_2 \in A_2 \cup M$ and any $s \in [0, 1]$, selecting a p-periodic solution u_i to $x' + q(t)x = f(t, x) + h_i(t)$, we have

$$\begin{aligned} &f(t, su_1 + (1-s)\, u_2) + sh_1 + (1-s)\, h_2 \\ &\quad \leqslant sf(t, u_1) + (1-s) f(t, u_2) + sh_1 + (1-s)\, h_2 \\ &\quad = su_1' + q(t)\, su_1 + (1-s)\, u_2' + q(t)(1-s)\, u_2. \end{aligned}$$

This means that $su_1 + (1-s)\, u_2$ is an upper solution of

$$x' + q(t)x = f(t, x) + sh_1 + (1-s)\, h_2, \qquad x\ p\text{-periodic}.$$

A lower solution α to this same problem with $\alpha \leqslant su_1 + (1-s)\,u_2$ can be constructed by taking a positive solution u to (L) and then setting $\alpha = -cu$ with $c > 0$ sufficiently large. Therefore $sh_1 + (1-s)\,h_2 \in A_2 \cup M$ by virtue of Lemma 0.2(b). In the case $(\partial^2/\partial x^2) f < 0$, we proceed in the same way using Lemma 0.2(a).

Now it remains to prove (A), (B), and (C). Set

$$l_1(t) = \min\{l_-(t), l_+(t)\}, \qquad l_2(t) = \max\{l_-(t), l_+(t)\}.$$

Ad (A). Let $(h_n)_n$ be a convergent sequence in C_p^0 and let $x_n = T^{-1}(h_n)$. We must show that $(x_n)_n$ has a convergent subsequence. To this aim, it suffices to show that $(x_n)_n$ is bounded. Suppose the contrary, i.e., that

$$\lim_n \|x_n\|_\infty = +\infty$$

by passing to a subsequence if necessary. We set $z_n = x_n/\|x_n\|_\infty$, $H_n(t) = h_n(t)/\|x_n\|_\infty$ and

$$p(t, x) = \int_0^1 f_x(t, \xi x)\, d\xi.$$

Dividing the equation

$$x_n' + q(t)\, x_n = f(t, x_n) + h_n(t)$$

(equivalent to $T(x_n) = h_n$) by $\|x_n\|_\infty$ we get

$$z_n' + q(t)\, z_n = p(t, x_n)\, z_n + H_n(t). \tag{6}$$

Since everything to the right of z_n' is bounded, by the Ascoli theorem there is a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that

$$\lim_k z_{n_k} = z$$

uniformly. Define $l \in L^\infty$ by

$$l(t) = \begin{cases} l_1(t) & \text{if } \ z(t) < 0 \\ 0 & \text{if } \ z(t) = 0 \\ l_2(t) & \text{if } \ z(t) > 0. \end{cases}$$

Obviously $\lim_k p(t, x_{n_k})\, z_{n_k} = l(t) z$ pointwise. Therefore, writing (6) with $n = n_k$ in the equivalent form

$$z_{n_k}(t) = z_{n_k}(0) + \int_0^t \{-q(s)\, z_{n_k}(s) + p(s, x_{n_k}(s))\, z_{n_k}(s) + H_{n_k}(s)\}\, ds$$

and passing to the limit we get

$$z(t)=z(0)+\int_0^t \{-q(s)+l(s)\}\, z(s)\, ds.$$

This means that z is a solution of the ordinary differential equation

$$u'+\{q(t)-l(t)\}\, u=0. \tag{7}$$

Since the null function is a solution to (7) and since (7) enjoys uniqueness for the Cauchy problems, z must have a constant sign. Therefore k reduces to l_- or l_+, and then it follows from Lemma 0.1 that $\lambda=0$ is not an eigenvalue of

$$u'+\{q(t)-l(t)\}\, u=\lambda u, \qquad u \ p\text{-periodic}.$$

Thus $z\equiv 0$, a contradiction.

Ad (B). By virtue of the formula (4) giving the derivative of T at u, we have $u_0 \in W$ if and only if the problem

$$x'+q(t)x-f_x(t, u_0(t))x=0, \qquad x \ p\text{-periodic} \tag{8}$$

has a nonnull solution. This is equivalent to saying that $\lambda=0$ is an eigenvalue of

$$x'+q(t)x-f_x(t, u_0(t))x=\lambda x, \qquad x \ p\text{-periodic}.$$

Then Ker $T'(u_0)$ is spanned by a nonzero vector v_0 and so hypothesis (I) of Theorem 2.7 in Ambrosetti and Prodi [2] is satisfied. To verify hypothesis (II*) of Theorem 2.7 of Ambrosetti and Prodi [2], we observe first that it follows from Lemma 0.1 or, alternately, from Section 8.15 in Chapter VI of Rouche and Mawhin [3] that the adjoint equation

$$-x'+q(t)x-f_x(t, u_0(t))x=0 \tag{9}$$

has nonnull p-periodic solutions since (8) does. Let w_0 be a nonnull p-periodic solution to (9) and define a functional γ_0 on C_p^0 by

$$\gamma_0(x)=\int_0^p x w_0\, dt.$$

It follows from Theorem 8.17 in Chapter VI of Rouche and Mawhin [3] that $\operatorname{Im} T'(u_0)=\operatorname{Ker} \gamma_0$. By (5), condition (II*) of Theorem 2.7 in Ambrosetti and Prodi [2] becomes, in our case,

$$\gamma_0(T''(u_0)\cdot v_0\cdot v_0)=\int_0^p \frac{\partial^2}{\partial x^2} f(t, u_0(t))\cdot v_0^2(t)\, w_0(t)\, dt.$$

By uniqueness of Cauchy problems, v_0 and w_0 have a constant sign. Therefore

$$\gamma_0(T''(u_0)\cdot v_0\cdot v_0)\neq 0$$

and condition (II*) of Theorem 2.7 in Ambrosetti and Prodi [2] holds. We can conclude that every $u_0\in W$ is an ordinary singular point. To complete the proof of (B), we must show that W is not empty and connected. To this aim, we shall show that W is a continuous image of a linear subspace of C_p^0 with codimension 1. Let u_1 be the constant function $u_1(t)\equiv 1$, and let Z be the topological supplement in C_p^0 to the linear subspace $\mathbb{R}u_1=\mathrm{sp}(u_1)$ spanned by u_1: every $x\in C_p^0$ can be represented in a unique way in the form

$$x=z+su_1$$

with $z\in Z$ and $s\in\mathbb{R}$. For each $z\in Z$ and $s\in\mathbb{R}$, let $\lambda_z(s)$ be the unique eigenvalue of

$$x'+q(t)x-f_x(t,z(t)+su_1(t))x=\lambda x, \qquad x\ p\text{-periodic.}$$

By Lemma 0.1, $\lambda_z(s)$ depends continuously on z and s jointly. Choose $z\in Z$. The assumptions of the theorem guarantee the existence of two constants a, b with $a<0<b$ and

$$f_x(t,z(t)+au_1(t))<\lambda_q<f_x(t,z(t)+bu_1(t)) \qquad (0\leqslant t\leqslant p) \tag{10}$$

or alternately

$$f_x(t,z(t)+au_1(t))>\lambda_q>f_x(t,z(t)+bu_1(t)) \qquad (0\leqslant t\leqslant p). \tag{11}$$

Assume (10) since (11) can be treated similarly. From (10) and the comparison property of eigenvalues expressed by Lemma 0.1 we get

$$\lambda_z(a)>\lambda_{q-\lambda_q}>\lambda_z(b),$$

where $\lambda_{q-\lambda_q}$ is the eigenvalue of

$$x'+q(t)x-\lambda_q x=\lambda x, \qquad x\ p\text{-periodic.}$$

Obviously $\lambda_{q-\lambda_q}=0$, hence we have

$$\lambda_z(a)>0>\lambda_z(b).$$

Since $f_x(t,\cdot)$ is strictly monotone, the function $\lambda_z(\cdot)$ is strictly monotone by the comparison property of eigenvalues given by Lemma 0.1. Moreover $\lambda_z(\cdot)$ is continuous as we have seen. Therefore there exists a unique $s_z\in\,]a,b[$ such that

$$\lambda_z(s_z)=0.$$

As we have noted above, this means that $z+s_z u_1 \in W$. Every $x \in W$ can be represented in this way: given $x \in W$, there are $z_x \in Z$ and $\alpha_x \in \mathbb{R}$ such that $x = z_x + \alpha_x u_1$; as we have just seen, in correspondence to z_x there is a unique $s_{z_x} \in \mathbb{R}$ such that $z_x + s_{z_x} u_1 \in W$; therefore $\alpha_x = s_{z_x}$. We may conclude that the mapping $\Psi: Z \to W$ defined by $\Psi(z) = z + s_z u_1$ is a bijection. Now we want to show that Ψ is continuous. Assume $\lim_n z_n = z_0$ in Z. Since $\{z_n \mid n \geqslant 1\}$ is bounded, we find a, b such that $a<0<b$ and (10) or (11) hold for $z = z_n$, all $n \geqslant 1$. Then

$$a < s_{z_n} < b \qquad (n \geqslant 1)$$

and so for every $n_k \uparrow \infty$ there is $n_{k_i} \uparrow \infty$ such that

$$\lim_i s_{z_{n_{k_i}}} = s_\infty$$

for a suitable $s_\infty \in [a, b]$. Since $\lambda_z(s)$ depends continuously on (z, s), we have

$$\lim_i \lambda_{z_{n_{k_i}}}(s_{z_{n_{k_i}}}) = \lambda_{z_0}(s_\infty)$$

and so $s_\infty = s_{z_0}$. This implies that

$$\lim_m s_{z_n} = s_{z_0}$$

and therefore Ψ is continuous. It follows that W is a nonempty, connected set. W is closed by virtue of the continuity of Ψ and the fact that if $\lim_n z_n + s_n u_1 = y$, then $(z_n)_n$ and $(s_n)_n$ converge respectively to the first and second coordinates of y.

Ad (C). Suppose that $u_0 \in W$ and that there is $u \neq u_0$ such that $T(u) = T(u_0)$. Setting

$$g(t) = \begin{cases} \dfrac{f(t, u(t) - f(t, u_0(t)}{u(t) - u_0(t)} & \text{if} \quad u(t) \neq u_0(t) \\ f_x(t, u_0(t)) & \text{if} \quad u(t) = u_0(t) \end{cases}$$

it is easily seen that $u - u_0$ is a nonnull solution of

$$x' + (q(t) - g(t))x = 0. \tag{12}$$

Since $x_0 \equiv 0$ is a solution to this equation, it follows from the uniqueness of Cauchy problems that $u - u_0$ has constant sign. Therefore we have

$$g > f_x(\cdot, u_0(\cdot)) \qquad \text{or alternatively,} \qquad g < f_x(\cdot, u_0(\cdot)).$$

Then the comparison property for eigenvalues expressed in Lemma 0.1 implies that the unique eigenvalue λ' of

$$x' + (q(t) - g(t))x = \lambda x, \qquad x \ p\text{-periodic}$$

is smaller or, alternatively, greater than the unique eigenvalue λ'' of

$$x' + (q(t) - f_x(t, u_0(t)))x = \lambda x, \qquad x \text{ } p\text{-periodic.}$$

Since $u - u_0$ is a nonnull solution to (12), we have $\lambda' = 0$. Since $u_0 \in W$, we have $\lambda'' = 0$ by the remarks at the beginning of the proof of (B). Therefore we have $0 \lessgtr 0$, a contradiction which proves (C). Q.E.D.

Remark. Theorems 1 and 2 show that if $(\partial/\partial x) f$ crosses λ_q in a suitable way, then we have more than one solution. An argument similar to the one used to prove Lemma 2 shows that if $(\partial/\partial x) f$ stays above or below λ_q, i.e., if one of the following conditions holds:

(i) $(\partial/\partial x) f \leqslant \lambda_q$ (resp. $(\partial/\partial x) f \geqslant \lambda_q$) with equality for almost all x whenever t is fixed; or

(ii) $(\partial/\partial x) f \leqslant h$ (resp.: $(\partial/\partial x) f \geqslant h$) with $h \in L^1$ and $h \leqslant \lambda_q$ (resp. $h \geqslant \lambda_q$) with strict inequality in a set of positive measure;

then (P) has at most one solution.

References

1. A. Ambrosetti and G. Mancini, Sharp nonuniqueness results for some nonlinear problems, *Nonlinear Anal.* **3** (1979), 635–645.
2. A. Ambrosetti and G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, *Ann. Mat. Pura Appl.* **93** (1972), 231–246.
3. N. Rouche and J. Mawhin, "Equations différentielles ordinaires," Vol. I, Masson, Paris 1973.
4. G. Vidossich, "Existence and Uniqueness Results for Boundary Value Problems from the Comparison of Eigenvalues," ICTP Internal Report IC/79/15, January 1979.